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The properness of weak stability notions☆

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ABSTRACT

Multiple weak stability notions have been proposed for two-sided matching problems, each based on a different perspective to justify the existence of certain blocking pairs. We introduce a criterion, called *properness*, to evaluate these weak stability notions. We say a notion of weak stability is proper if, whenever a matching is weakly stable, every matching that is more stable than it (admitting a subset of blocking pairs) is also weakly stable. We show that among existing notions of weak stability: the von Neumann and Morgenstern stable set, reasonable stability, α-equitability, partial stability, and the Tang-Zhang weak stability are proper, whereas the Klijn-Massó weak stability, essential stability, and priority-neutrality are not proper.

1. Introduction

This paper focuses on two-sided matching markets, including one-to-one matching markets (e.g., the marriage market as in Gale and Shapley, 1962) and many-to-one matching markets (e.g., the school choice problem as in Abdulkadiroglu and Sönmez, 2003). A matching market outcome, which is called a matching, is a set of partnerships for all agents. A matching μ is **individually rational** if no agent prefers being unmatched over the assignment under μ. Gale and Shapley (1962) define an individually rational matching μ as **stable** if there exists no blocking pair—a pair of agents from both sides that prefer each other to their assignments under μ.[1]

Stability is a fundamental concept in two-sided matching markets. In the literature, various weak stability notions have been introduced to weaken stability, motivated primarily by two considerations:

1. Theoretically, a blocking pair may need to provide reasonable arguments, such as proposing an alternative matching that satisfies certain properties, to justify blocking.
2. In school choice problems, the matching produced by Gale and Shapley (1962)'s deferred acceptance (DA) algorithm, i.e., the student-optimal stable matching, is often Pareto inefficient (Roth, 1982; Abdulkadiroglu and Sönmez, 2003).[2] Pareto improving on this matching inevitably violates stability, and weak stability notions are important for guiding the selection of a Pareto improvement.

Influential weak stability notions in the literature include the von Neumann and Morgenstern (vNM) stable set (von Neumann and Morgenstern, 1944; Ehlers, 2007), the Klijn-Massó weak stability (Klijn and Massó, 2003), reasonable stability (Kesten, 2004), α-equitability (Alcalde and Romero-Medina, 2017), partial stability (Dur et al., 2019), essential stability (Troyan et al., 2020), the Tang-Zhang weak stability (Tang and Zhang, 2021), and priority-neutrality (Reny, 2022), all developed from different perspectives to support the presence of certain blocking pairs.

We introduce the criterion of 'properness' to evaluate weak stability notions. Following Abdulkadiroglu et al. (2020), we say a matching μ is more stable than another matching ν if every blocking pair of μ is also a blocking pair of ν. A weak stability notion is proper if, for each school choice problem, whenever a matching μ is weakly stable, every matching more stable than μ is also weakly stable.

We show that (1) the vNM stable set, reasonable stability, α-equitability, partial stability (with respect to any set of acceptable priority violations), and the Tang-Zhang weak stability are proper, while (2) the Klijn-Massó weak stability, essential stability, and priority-neutrality are not proper.

This paper is organized as follows: Section 2 introduces the model; Section 3 presents the main result; and finally, Section 4 provides the conclusion.

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[1] We defer a more formal definition to Section 2.

[2] DA achieves Pareto efficiency only if the priority structure satisfies the conditions specified by Ergin (2002). Abdulkadiroglu et al. (2009) show that welfare losses for students under DA are significant in New York City high school admissions.

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2. Preliminaries

A set of students I are to be assigned seats at a set of schools S or the null school $\varnothing$. Each school $s \in S$ has a finite number of available seats, or quota, q_s. The null school $\varnothing$ has no quota limit, which permits any student to remain unmatched.

Each student i holds a strict preference P_i over the set $S \cup \{\varnothing\}$, and we write $s\ R_i\ s'$ to mean $s\ P_i\ s'$ or $s = s'$. Each school s has a strict priority list over the set I.[3] A school $s \in S$ is acceptable to a student $i \in I$ if $s\ P_i\ \varnothing$ and unacceptable if $\varnothing\ R_i\ s$.

A school choice problem consists of a pair $(P, \succ)$, where $P := (P_i)_{i \in I}$ denotes the students' preferences and $\succ := (\succ_s)_{s \in S}$ denotes the schools' priority lists.

A **matching** μ is a mapping from I to $S \cup \{\varnothing\}$ such that $|\mu^{-1}(s)| \leq q_s$ for all $s \in S$.

A pair $(i, s) \in I \times S$ is a **blocking pair** of a matching μ if: (1) student i's priority is violated by another student j: $s\ P_i\ \mu(i)$ and $i \succ_s j$ with $\mu(j) = s$; or (2) μ is wasteful: $s\ P_i\ \mu(i)$ and $|\mu^{-1}(s)| < q_s$. Let $B(\mu)$ denote the set of all blocking pairs for μ.

A matching μ is **individually rational** if for each student $i \in I$ we have $\mu(i)\ P_i\ \varnothing$. An individually rational matching μ is **stable** if there does not exist any blocking pair.

A matching μ **Pareto dominates** another matching ν if each student (weakly) prefers the assignment under μ to that under ν, i.e., if $\nu \neq \mu$ and $\mu(i) R_i \nu(i)$ for each $i \in I$. A matching is **Pareto efficient** if it is not Pareto dominated by any other matching.

3. Properness of weak stability notions

3.1. Definition of properness

A **weak stability notion** is a correspondence that maps every school choice problem to a set of matchings that includes all stable matchings.

According to Abdulkadiroglu et al. (2020), a matching ν is **more stable** than another matching μ if $B(\nu) \subseteq B(\mu)$. In this case, we also say that μ is **less stable** than ν. It is natural to require that if a matching is weakly stable, any matching more stable than it should also be weakly stable. Based on this idea, we propose a criterion termed 'properness' to evaluate weak stability notions.

Definition 1 (*Properness*)**.** A weak stability notion is **proper** if, for each school choice problem, whenever a matching μ is weakly stable with respect to this notion, every matching ν that is more stable than μ is also weakly stable with respect to it.

The properness of a weak stability notion $\mathcal{A}$ means that if no blocking pair for a matching μ can justify their blocking (as required by $\mathcal{A}$), then no blocking pair for any more stable matching ν can do so either.

Note that a weak stability notion is proper does not imply that every stricter notion is also proper. When we refine a proper weak stability notion, the refined notion is proper if and only if any matching that is less stable than an excluded matching is also excluded.

Likewise, when we weaken a proper notion, the new notion is proper if and only if any matching that is more stable than a newly included matching is also included. These two conditions can be used to verify the properness of weak stability notions that are not covered in this paper.[4]

[3] It is assumed that each school considers all students acceptable.

[4] For instance, let $\mathcal{A}$ be the set of matchings that are more stable than EADA (where all students consent to all schools), then $\mathcal{A}$ is proper because the Tang-Zhang weak stability is proper according to Theorem 1.

3.2. Notions of weak stability

The following important weak stability notions have been proposed in the literature, motivated by the need for blocking pairs to provide reasonable arguments or the trade-off between stability and efficiency in the context of school choice. All of these notions are defined exclusively for individually rational matchings.

(i) **The vNM stable set**: The **vNM stable set** is introduced by von Neumann and Morgenstern (1944) for cooperative games, and it is first studied by Ehlers (2007) in marriage market.[5] For a school choice problem, the vNM stable set V contains precisely every individually rational matching such that none of its blocking pairs can be matched by any matching in the set.
By definition, the vNM stable set is proper: if a matching μ is in V, no pair in $B(\mu)$ can be matched by any matching in V, which implies any matching ν more stable than μ is also in V.

(ii) **Reasonable stability**: Kesten (2004) defines a matching as **reasonably stable** if none of its blocking pairs can be matched by a stable matching.
By definition, reasonable stability is proper: for any matching ν that is more stable than a reasonably stable matching μ, no pair in $B(\nu) \subseteq B(\mu)$ can be matched by a stable matching, which implies that ν is also reasonably stable.

(iii) **α-equitability**: Alcalde and Romero-Medina (2017) define a matching as α-equitable if no student can propose an alternative matching that no other student might oppose. Formally, a matching μ is **α-equitable** if, for each pair $(i, s) \in B(\mu)$, every matching μ' that assigns i to s must violate the priority of some other student $j \neq i$.
By definition, α-equitability is proper: for any matching ν that is more stable than an α-equitable matching μ, every pair $(i, s) \in B(\nu) \subseteq B(\mu)$ satisfies that any matching μ' assigning i to s must violate the priority of some other student j, which implies ν is also α-equitable.

(iv) **Partial stability** (**with respect to** C): Dur et al. (2019) introduce partial stability by allowing certain priority violations. Formally, for any set of acceptable priority violations C (a set of student-school pairs), a matching μ is **partially stable with respect to** C if $B(\mu) \subseteq C$.
By definition, partial stability (with respect to C) is proper: for any matching ν that is more stable than a partially stable (with respect to C) matching μ, we have $B(\nu) \subseteq B(\mu) \subseteq C$, which implies ν is also partially stable (with respect to C).

(v) **The Tang-Zhang weak stability**: Tang and Zhang (2021) define a matching μ as **weakly stable** if none of its blocking pairs can be matched by a more stable matching μ' (i.e., a matching μ' such that $B(\mu) \subseteq B(\mu)$).
By definition, the Tang-Zhang weak stability is proper: for any matching ν that is more stable than a weakly stable matching μ, all pair in $B(\nu) \subseteq B(\mu)$ cannot be matched by any matching μ' such that $B(\mu') \subseteq B(\nu) \subseteq B(\mu)$, which implies ν is also weakly stable.

(vi) **The Klijn-Massó weak stability**: To adapt (Zhou, 1994)'s bargaining set to marriage market, Klijn and Massó (2003) define a matching as weakly stable if, for each of its blocking pairs, one partner can find a more attractive match to form another blocking pair.[6] Formally, a matching μ is **Klijn-Massó weakly stable** if, for each of its blocking pairs $(i, s) \in B(\mu)$:

[5] Ehlers and Morrill (2020) develop the notion of the legal set by extending the vNM stable set to school choice problems where schools are associated with possibly non-responsive choice functions.

[6] Klijn and Massó (2003) show that the bargaining set in the marriage market corresponds to the set of matchings that are both Klijn-Massó weakly stable and Pareto efficient. Klijn and Massó (2003) introduce the Klijn-Massó weak stability for the marriage market (Gale and Shapley, 1962), and we extend it to the school choice problem.

- The student i can find a more desirable school s' that also violates her priority in μ; or
- The school s can find q_s other students with higher priority than i, each of whom either is assigned to s by μ or forms a blocking pair with s for μ.

(vii) **Essential stability**: Troyan et al. (2020) define a matching μ as **essentially stable** if, for any blocking pair $(i, s) \in B(\mu)$, the following reassignment chain renders i's claim to s vacuous:

- If i claims a seat at s, excluding the lowest-priority student j admitted, j will then claims her most preferred school willing to block with her, excluding another student.
- This process repeats until a student claims s and excludes i.

(viii) **Priority-neutrality**: Reny (2022) defines a matching as **priority-neutral** if it is not possible to make any student whose priority is violated better off without violating the priority of some student who is made worse off.

3.3. Result

Our main result is as follows.

Theorem 1. *Among existing notions of weak stability, (1) the vNM stable set, reasonable stability, α-equitability, partial stability (with respect to C), and the Tang-Zhang weak stability are proper, whereas (2) the Klijn-Massó weak stability, essential stability, and priority-neutrality are not.*

As shown in Section 3.2, statement (1) follows directly from the definitions of the respective notion of weak stability. Under each weak stability notion among the vNM stable set, reasonable stability, α-equitability, partial stability, and the Tang-Zhang weak stability, if a matching ν is more stable than a matching μ, then it is weakly easier to justify the existence of any blocking pair in $B(\nu) \subseteq B(\mu)$ under ν than under μ. Such an observation is sufficient for showing that a notion of weak stability is proper.

However, no unified argument can be used to show statement (2), that certain notions of weak stability are not proper. Different notions of weak stability fail to be proper due to different reasons. In general, we should be alerted about a notion's properness if under this notion, justifying the existence of a blocking pair under a matching μ relies on information beyond $B(\mu)$. For example, we should be alerted that the definition of priority neutrality pays attention to students who become worse off at the new matching proposed to block μ. Fortunately, to prove that a weak stability notion is not proper, it suffices to identify a single school choice problem where it fails. Therefore, we prove statement (2) by providing a counterexample.

Proof. We have already proven statement (1) in Section 3.2, let us now prove statement (2) using a counterexample. The table below provides a school choice problem $(P, \succ)$, where students in $I = \{i_1, i_2, i_3, i_4, i_5\}$ are to be assigned to schools in $S = \{s_1, s_2, s_3, s_4, s_5\}$, each with one available seat.

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
s_2	s_3	s_2	s_1	s_4	i_1	i_2	i_3	i_4	i_5
s_1	s_1	s_4	s_3	s_5	i_2	i_3	i_4	i_5	i_2
$\varnothing$	s_2	s_3	s_4	$\varnothing$	i_4	i_1	i_2	i_3	$\vdots$
	s_5	$\varnothing$	$\varnothing$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	$\varnothing$								

The table below presents three candidate matchings and their blocking pairs.

	s_1	s_2	s_3	s_4	s_5	Blocking pairs
μ_1	i_4	i_1	i_2	i_3	i_5	$(i_3, s_2), (i_5, s_4)$
μ_2	i_2	i_1	i_3	i_4	i_5	(i_3, s_2)
μ_3	i_4	i_1	$\varnothing$	i_5	i_2	$(i_2, s_1), (i_2, s_2), (i_2, s_3), (i_3, s_2), (i_3, s_3)$

The matching μ_1 is the result of the efficiency-adjusted deferred acceptance mechanism (EADAM) (Kesten, 2010; Tang and Yu, 2014) with all students' consent.[7] In the first round, i_5's consent at s_5 leads to an exchange between i_3 and i_4. In the second round, i_3 becomes unimprovable, and her consent at s_4 triggers an exchange among i_1, i_2, and i_3. Thus, μ_1 is essentially stable by Theorem 1 in Troyan et al. (2020) and priority-neutral by Theorem 2 in Reny (2022).

However, the matching μ_2, which is more stable than μ_1, is neither essentially stable nor priority-neutral. This is because: firstly, for the pair $(i_3, s_2) \in B(\mu_2)$, student i_3's claim for school s_2 is not vacuous, as shown by the reassignment chain: $i_3 \to s_2 \to i_1 \to s_1 \to i_2 \to s_3$; secondly, the matching μ_1 can make student i_3, whose priority is violated by μ_2, better off without violating the priority of any student who is made worse off (no student is made worse off in this case).

Therefore, both essential stability and priority-neutrality are not proper.

The matching μ_3 is Klijn-Massó weakly stable because, for every blocking pair, one of the members can find a more attractive partner to form another pair in $B(\mu_1)$: $(i_3, s_2) \xrightarrow{s_2} (i_2, s_2) \xrightarrow{i_2} (i_2, s_3) \xrightarrow{s_3} (i_3, s_3) \xrightarrow{i_3} (i_3, s_2)$: and $(i_2, s_1) \xrightarrow{i_2} (i_2, s_3)$. However, the matching μ_2, which is more stable than μ_3, is not Klijn-Massó weakly stable, since for the pair $(i_3, s_3) \in B(\mu_2)$ both i_3 and s_3 cannot find a more attractive partner to form another pair in $B(\mu_2)$. Therefore, the Klijn-Massó weak stability is not proper.

We have completed the proof. □

4. Conclusion

Various notions of weak stability have been proposed to facilitate blocking pairs reaching agreements and to address the trade-off between stability and students' welfare in school choice. With numerous notions of weak stability in the literature, a natural question arises: how should we compare and identify the most desirable ones? Following Abdulkadiroglu et al. (2020), we consider a matching μ to be more stable than another matching ν if $B(\mu) \subseteq B(\nu)$. Based on this idea, we propose the criterion – properness – for evaluating the notions of weak stability.

Although properness is a binary criterion that classifies weak stability notions as either proper or not, it provides a rigorous framework for evaluating these notions and facilitates their systematic comparison. However, while properness offers a valuable perspective for comparison, it does not encompass all relevant characteristics of certain notions. For instance, many weak stability notions aim to balance stability with practical criteria such as efficiency. Thus, labeling a weak stability notion as 'not proper' does not imply that the motivations behind its proposal are invalid.[8]

One potential direction for future research is to introduce a notion of partial properness or degree of properness to provide a more detailed evaluation of weak stability notions.[9] Another direction is to define alternative notions of properness based on different definitions of the 'more stable than' relation. To study the robustness of TTC's maximal stability, Doğan and Ehlers (2022) have introduced a unified framework to compare the instability of matchings. Accordingly, their work induces a unified approach to defining the properness of weak stability notions.

Data availability

No data was used for the research described in the article.

[7] EADAM is proposed by Kesten (2010) to recover students' efficiency loss in DA. Tang and Yu (2014) simplify it by iteratively removing Pareto unimprovable students.

[8] We thank an anonymous referee for pointing this out.

[9] We thank an anonymous referee for suggesting this direction of future research.

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