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Some new characterizations of top trading cycles☆

Yajing Chen [a,*], Zhenhua Jiao [b], Chenfeng Zhang [a], Luosai Zhang [c]

[a] School of Business, East China University of Science and Technology, Shanghai, 200237, China
[b] School of Business, Shanghai University of International Business and Economics, Shanghai, 201620, China
[c] Center for Economic Research, Shandong University, Jinan, 250199, China

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ABSTRACT

This paper studies the housing market problem introduced by Shapley and Scarf (1974). We refine some existing characterizations of the well-known top trading cycles (TTC) rule by relaxing strategy-proofness and efficiency. Specifically, our main result shows that a rule is equivalent to the TTC rule if and only if it satisfies either of the following two sets of axioms: (1) individual rationality, pair-efficiency, and upper invariance; and (2) individual rationality, endowments-swapping-proofness, and upper invariance. Additionally, we clarify the logical relationships among a set of axioms that are weaker than strategy-proofness. New characterizations of the TTC rule are obtained as direct consequences of our main theorems.

1. Introduction

This paper considers the classical housing market problem introduced by Shapley and Scarf (1974). In this setting, there is a set of agents, each of whom initially owns one object and has strict preferences over all objects. An allocation is defined as a permutation of the objects among the agents. A rule is a systematic procedure that finds an allocation for every possible combination of initial endowments and preference profiles.

The top trading cycles (TTC) rule selects an allocation via the TTC algorithm, which was first introduced by Shapley and Scarf (1974). In each round of the TTC algorithm, agents point to their favorite remaining objects and objects point to their owners. In this way, there exists at least one cycle consisting of agents and objects. Each agent in a cycle receives the object she points to, and is then removed together with the object she received. The process repeats until all agents and objects are removed. The TTC rule has been extensively studied in the literature. This is mainly because the TTC allocation coincides with the core (under weak domination) (Roth and Postlewaite, 1977) and satisfies desirable properties such as individual rationality, Pareto efficiency, strategy-proofness (Roth, 1982), and group strategy-proofness (Bird, 1984).[1]

We say the TTC rule is *characterized* by a set of axioms if it is the only rule that satisfies all the specified conditions. The existing literature on characterizing the TTC rule can be classified into three categories as follows.[2] The first category consists of characterizations based on group strategy-proofness. Notable examples include Sönmez (1996)'s characterization by Nash implementability, individual rationality, and Pareto efficiency and Takamiya (2001)'s characterization by strategy-proofness, non-bossiness, individual rationality, and ontoness. Miyagawa (2002) characterized both the TTC rule and the no-trade rule by individual rationality, strategy-proofness, non-bossiness, and anonymity.

The characterizations in the second category primarily employ strategy-proofness. Ma (1994) characterized the TTC rule using

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* Corresponding author.
E-mail addresses: yajingchen@ecust.edu.cn (Y. Chen), zhjiao@suibe.edu.cn (Z. Jiao), zhangluosai@sdu.edu.cn (L. Zhang).

[1] Sandholtz and Tai (2024) proposed a critical claim and provided a corrected proof for Bird (1984).

[2] Several papers characterized much broader classes of rules that include the TTC rule such as Pápai (2000), Miyagawa (2002), Tang and Zhang (2016), and Pycia and Ünver (2017).

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individual rationality, Pareto efficiency, and strategy-proofness.[3] Fujinaka and Wakayama (2018) characterized it with individual rationality, strategy-proofness, and endowments-swapping-proofness. Chen and Zhao (2021) established two alternative characterizations: one based on individual rationality, Pareto efficiency, and rank monotonicity, and the other based on individual rationality, endowments-swapping-proofness, and rank monotonicity. Ekici (2024) further refined Ma (1994)'s characterization by relaxing Pareto efficiency to pair-efficiency while preserving individual rationality and strategy-proofness.[4] More recently, Klaus et al. (2025) introduced a characterization based on pair-efficiency, respecting-improvement, and strategy-proofness. This is the first characterization that does not rely on individual rationality.

The third category consists of characterizations that refine results in the first and second categories by relaxing strategy-proofness. Altuntaş et al. (2023) refined (Ma, 1994)'s result and characterized the TTC rule through the following two groups of axioms: (1) individual rationality, Pareto efficiency and drop strategy-proofness; and (2) individual rationality, Pareto efficiency and upper invariance.[5] Coreno and Feng (2024) refined Ma (1994)'s characterization by replacing strategy-proofness with endowment-truncation-proofness while maintaining individual rationality and Pareto efficiency. Similarly, Coreno and Feng (2025) refined Fujinaka and Wakayama (2018)'s characterization by weakening strategy-proofness to endowment-truncation-proofness while maintaining individual rationality and endowments-swapping-proofness.[6]

In this spirit of characterizations in the third category, we relax strategy-proofness to upper invariance and obtain two new characterization results. Strategy-proofness of a rule requires that no agent can be strictly better off by misreporting her preferences. Upper invariance of a rule means that an agent's assignment remains unchanged when she reports a preference that only modifies the ranking of objects below her current allocation.

Our first result (Theorem 1) characterizes the TTC rule through individual rationality, pair-efficiency, and upper invariance. Individual rationality requires that every agent weakly prefers her assignment to her endowment, and pair-efficiency requires that no pair of agents can strictly benefit from a bilateral exchange of their assigned objects. Our second result (Theorem 2) characterizes the TTC rule by individual rationality, endowments-swapping-proofness, and upper invariance. A rule satisfies endowments-swapping-proofness if no pair of agents can strictly gain by exchanging their endowments before the rule is applied. Our Theorem 1 refines the characterization in Ekici (2024), and Theorem 2 refines the result in Fujinaka and Wakayama (2018). As a by-product, Theorem 1 also improves upon the characterization in Altuntaş et al. (2023) by replacing Pareto efficiency with pair-efficiency.

The new characterizations deepen our understanding of the TTC rule. In order to prove the stronger characterization results, we introduce new analytical methods. These help us get a clearer understanding of TTC rule. Our characterizations implies that, under a set of natural and desirable requirements such as individual rationality, pair-efficiency, and upper invariance, TTC emerges as the uniquely admissible rule. This further underscores the theoretical prominence of the TTC rule. Characterizing the TTC rule with weaker axioms enhances its practical applicability, making it a more appealing choice than alternative rules.

The paper is organized as follows. Section 2 introduces the basic model, relevant axioms and the TTC rule. Section 3 presents the characterization of the TTC rule using individual rationality, pair-efficiency, and upper invariance. Section 4 presents the characterization of the TTC rule using individual rationality, endowments-swapping-proofness, and upper invariance. Section 5 discusses the logical relationships among several axioms that are weaker than strategy-proofness and provides several new characterizations of the TTC rule. Section 6 presents a short conclusion.

2. Preliminaries

2.1. Model

Let $N \equiv \{1, 2, \ldots, n\}$ be the set of **agents** with $n \geq 2$. Each agent i is endowed with ω_i, and hence the set of available objects is $H = \{\omega_1, \ldots, \omega_n\}$.

Each agent $i \in N$ has a strict **preference** relation P_i over objects in H. Let R_i be the weak preference relation induced from P_i, such that hR_ih' if and only if either hP_ih' or $h = h'$. An object h is said to be **acceptable** to agent i at preference P_i if $hR_i\omega_i$. For each $i \in N$, $P_i \in \mathcal{P}$, and $h \in H$, let $P_i(h)$ be the preference **rank** of object h at P_i, that is, for each $k \in \{1, 2, \ldots, n\}$, $P_i(h) = k$ means that the object h is the kth preferred object in agent i's preference relation P_i. Let $\mathcal{P}$ be the set of all strict preferences over H. Let $P \equiv (P_1, P_2, \ldots, P_n) \in \mathcal{P}^n$ be the preference profile of all agents. For a subset $N' \subseteq N$, the preference profile of agents in N' is denoted by $P_{N'}$. Given $i \in N$, $P_i' \in \mathcal{P}$ and $P \in \mathcal{P}^n$, let $-i$ be $N \backslash \{i\}$, and (P_i', P_{-i}) be the preference profile where only P_i is replaced with P_i'.

For each $P_i \in \mathcal{P}$ and $h \in H$, let the **upper contour set** and **lower contour set** of P_i at h be $U(P_i, h) = \{h' \in H : h'R_ih\}$ and $L(P_i, h) = \{h' \in H : hR_ih'\}$, respectively. Let the **strict upper contour set** and **strict lower contour set** of P_i at h be $\hat{U}(P_i, h) = \{h' \in H : h'P_ih\}$ and $\hat{L}(P_i, h) = \{h' \in H : hP_ih'\}$, respectively. For each $P_i \in \mathcal{P}$ and $H' \subseteq H$, let $P_i|_{H'}$ be agent i's preference restricted to objects in H'. Given $P_i \in \mathcal{P}$ and $h, h' \in H$, we say that h' is **adjacent** to h at P_i if $|P_i(h') - P_i(h)| = 1$.

An **allocation** is a bijection $\mu : N \to H$, which assigns object $\mu(i)$ to every agent i. For convenience, we denote $\mu(i)$ by μ_i throughout the paper. For any subset $N' \subseteq N$, we also use $\mu_{N'}$ to indicate the set of objects assigned to agents in N' under allocation μ. Let $\mathcal{A}$ be the set of all possible allocations. Let $\omega \equiv (\omega_i)_{i\in N} \in \mathcal{A}$ denote an **endowment profile**. For each $N' \subseteq N$, we use $\omega_{N'}$ to denote the set of endowments belonging to agents in N'. An **economy** is a pair $e \equiv (P, \omega) \in \mathcal{P}^n \times \mathcal{A}$ consisting of the preference profile $P = (P_i)_{i\in N} \in \mathcal{P}^n$ and the endowment profile $\omega = (\omega_i)_{i\in N} \in \mathcal{A}$. Let $\mathcal{E} \equiv \mathcal{P}^n \times \mathcal{A}$ be the set of all possible economies. For any $N' \subseteq N$, an N'−allocation is a bijection $\mu^{N'} : N' \to \mu_{N'}$. Thus, an N−allocation is an allocation. An N'−allocation weakly dominates an allocation μ at an economy e if $\mu_i^{N'} R_i \mu_i$ for all $i \in N'$ and $\mu_j^{N'} P_j \mu_j$ for some $j \in N'$. The **core** of an economy consists of all allocations that are not weakly dominated by any N'−allocation.

A **rule** is a function $\varphi : \mathcal{E} \to \mathcal{A}$ that selects an allocation $\varphi(e) \in \mathcal{A}$ for each economy. For each $N' \subseteq N$, let $\varphi_{N'}(e)$ denote the set of objects allocated to agents in N' under the rule φ at the economy e.

2.2. Axioms

Individual rationality of a rule ensures no agent receives an assignment worse than her endowment. Formally, a rule φ satisfies **individual rationality (IR)** if for each economy $e = (P, \omega) \in \mathcal{E}$ and each agent $i \in N$, we have $\varphi_i(e)R_i\omega_i$.

[3] Svensson (1999) provided an alternative proof of Ma (1994)'s result, while H. (2015) and Sethuraman (2016) offered shorter proofs for this characterization.

[4] Ekici and Sethuraman (2024) provided a short proof for Ekici (2024)'s characterization result.

[5] Altuntaş et al. (2023)'s Corollary 5 characterized the TTC rule by individual rationality, Pareto efficiency and subset drop strategy-proofness. Subset drop strategy-proofness is strictly stronger than drop strategy-proofness, so the characterization by subset drop strategy-proofness follows directly from the characterization by drop strategy-proofness.

[6] Endowment-truncation-proofness is introduced as "truncation-proofness" in Coreno and Feng (2024, 2025).

An allocation is Pareto efficient if there does not exist another allocation in which at least one agent is strictly better off and no agent is strictly worse off. An allocation $\mu \in \mathcal{A}$ is Pareto efficient at $e = (P, \omega) \in \mathcal{E}$ if there does not exist another allocation $\mu' \in \mathcal{A}$ such that $\mu'_i R_i \mu_i$ for all $i \in N$ and $\mu'_j P_j \mu_j$ for some $j \in N$. A rule φ satisfies **Pareto efficiency (PE)** if it selects a Pareto efficient allocation $\varphi(e)$ for each economy $e = (P, \omega) \in \mathcal{E}$.

Pair-efficiency relaxes Pareto efficiency by requiring that no two agents can strictly improve through a bilateral exchange of their assigned objects. An allocation μ is pair-efficient at $e = (P, \omega) \in \mathcal{E}$ if there do not exist $i, j \in N$ such that $\mu_i P_j \mu_j$ and $\mu_j P_i \mu_i$. A rule φ satisfies **pair-efficiency (PaE)** if it selects a pair-efficient allocation $\varphi(e)$ for each economy $e = (P, \omega) \in \mathcal{E}$.

Endowments-swapping-proofness ensures that no pair of agents can strictly benefit from swapping their endowments before a rule is applied. For each $e = (P, \omega) \in \mathcal{E}$ and each pair of agents $i, j \in N$, let $e^{ij} = (P, \omega^{ij})$ denote the economy such that (1) $\omega^{ij}_i = \omega_j$, $\omega^{ij}_j = \omega_i$, and (2) for each $k \in N \backslash \{i, j\}$, $\omega^{ij}_k = \omega_k$. A rule φ satisfies **endowments-swapping-proofness (ESP)** if there does not exist any economy $e = (P, \omega) \in \mathcal{E}$ and any pair of agents $i, j \in N$ such that $\varphi_i(e^{ij}) P_i \varphi_i(e)$ and $\varphi_j(e^{ij}) P_j \varphi_j(e)$.

Strategy-proofness of a rule requires that no agent can be strictly better off by misreporting her preferences unilaterally. That is, truth-telling is a dominant strategy for each agent under that rule. Formally, a rule φ satisfies **strategy-proofness (SP)** if for each $e = (P, \omega) \in \mathcal{E}$, $i \in N$, and $e' = ((P'_i, P_{-i}), \omega) \in \mathcal{E}$, we have $\varphi_i(e) R_i \varphi_i(e')$.

Lower misrepresentations of an agent i at an object $h \in H$ are reports that i reveals the preference above h truthfully. Formally, given an preference P_i of an agent $i \in N$, a **lower misrepresentation** for P_i at $h \in H$ is a preference P'_i such that (1) $U(P'_i, h) = U(P_i, h)$, and (2) $P_i|_{U(P_i,h)} = P'_i|_{U(P_i,h)}$.

A rule satisfies upper invariance if, whenever an agent unilaterally reports a lower misrepresentation of her preference, her allocation remains unchanged.[7] Formally, a rule φ satisfies **upper invariance (UI)** if for each economy $e = (P, \omega) \in \mathcal{E}$ and each agent $i \in N$, the allocation of i remains unchanged under any unilateral lower misrepresentation P'_i at $\varphi_i(P, \omega)$, i.e.,

$$\varphi_i(P, \omega) = \varphi_i((P_i', P_{-i}), \omega).$$

2.3. The top trading cycles rule

The top trading cycles (TTC) rule, denoted by τ, is defined by specifying, for each economy $e = (P, \omega)$, an allocation $\tau(e)$ that is determined by the TTC algorithm. When running the TTC algorithm for an economy, a **cycle** is a sequence of distinct agents and objects such that each agent in the cycle points to her favorite object and each object points to its owner. Let C denote a generic cycle. A typical cycle C can be illustrated as follows:

$$i_1 \to \omega_{i_2} \to i_2 \to \omega_{i_3} \to \cdots \to i_{k-1} \to \omega_{i_k} \to i_k \to \omega_{i_{k+1}} \to i_{k+1} = i_1.$$

Here, $i_1, i_2, \ldots, i_k$ are distinct agents. Agent i_s points to object $\omega_{i_{s+1}}$, and $\omega_{i_{s+1}}$ points to its owner i_{s+1} for $s = 1, \ldots, k$.

The TTC algorithm works as follows:

Step 1. Each agent points to her favorite object. Each object points to its owner. Since there is a finite number of agents and objects, there exists at least one cycle. Assign each agent in each cycle the object to which she points and remove all agents and objects involved in any cycle from the economy.

Step t (t≥2). Each remaining agent points to her favorite object among the remaining ones. Each remaining object points to its owner. There exists at least one cycle. Assign each agent in each cycle the object to which she points and remove all agents and objects involved in any cycle from the remaining economy.

The algorithm terminates in a finite number of steps, when no agent and object remains.

Shapley and Scarf (1974) showed that the core is nonempty for each economy. Roth and Postlewaite (1977) further established that, in any economy, the unique allocation in the core coincides with the allocation produced by the TTC algorithm.

3. Characterizing via pair-efficiency

In this section, we introduce a novel characterization of the TTC rule. This characterization relies on the axioms of individual rationality, pair-efficiency, and upper invariance. Our result differs from the characterization in Ekici (2024) by replacing strategy-proofness with upper invariance. It also differs from the characterization in Altuntaş et al. (2023) by replacing Pareto efficiency with pair-efficiency.

Throughout this section, we will omit ω from the economy and denote an economy solely by the preference profile. Our proof for the characterization result primarily follows the TTC-similarity index method introduced by Ekici (2024). Before presenting our theorem and proof, we introduce the TTC-similarity index.

For each economy $P \in \mathcal{P}^n$, let $C_t(P)$ denote the set of cycles obtained at step t of the TTC procedure. Given a cycle C, we say that an allocation μ **executes** C if for each agent i involved in this cycle, μ_i is the object to which she points in C, and say that μ **does not execute** C otherwise.

We define the TTC-similarity index ρ^φ for an arbitrary rule φ as a function which measures the similarity between the allocations assigned by φ and the TTC rule τ for each economy.

Definition 1. The **TTC-similarity index** is a function $\rho^\varphi : \mathcal{P}^n \to \mathbb{Z}_{++}$ such that

$$\rho^\varphi(P) = \begin{cases} \min\{t \mid \text{there exists } C \in C_t(P) \text{ such that} \\ \qquad\qquad \varphi(P) \text{ does not execute } C\}, & \text{if } \varphi(P) \neq \tau(P), \\ \qquad\qquad\qquad \infty, & \text{otherwise.} \end{cases}$$

Remark 1. The TTC similarity index defined in the current paper is different from Ekici (2024)'s original version in two aspects. First, our index depends only on the step of the algorithm while Ekici (2024)'s index depends both the step of the algorithm and the size of a cycle. Second, our index relies on the classical description of the TTC algorithm which eliminates all emerging cycles in a certain step, while Ekici (2024)'s index relies on a new description of the TTC algorithm, which specifies a unique cycle with the smallest size and eliminates only one such cycle in each step.

Theorem 1. *A rule satisfies individual rationality, pair-efficiency, and upper invariance if and only if it is the TTC rule.*

Proof. It is obvious that τ satisfies individual rationality, pair-efficiency, and upper invariance.

Next, we prove the "only if" part. Suppose there is a rule φ that differs from the TTC rule τ but meets the requirements of individual rationality, pair-efficiency, and upper invariance. The proof is conducted by contradiction. We first derive the allocation outcomes of φ under two specific preference structures. Then, by modifying the preferences of agents in the second preference structure, we demonstrate that the resulting allocation under the first preference structure differs from the previously established result, thereby leading to a contradiction. The proof of Lemma 1 below can be obtained by substituting upper invariance for strategy-proofness in the corresponding part of Ekici (2024)'s proof. The rest of the proof is original.

[7] Upper invariance was first introduced in Altuntaş et al. (2023) in a more general setting. In the classical housing market setting, their definition coincides with the one we adopt in this paper.

Let $\overline{\mathcal{P}}$ denote the set of preference profiles that **minimize** the TTC-similarity index of the rule φ, that is,

$$\overline{\mathcal{P}} = \arg\min_{P \in \mathcal{P}^n} \rho^{\varphi}(P).$$

For each $\overline{P} \in \overline{\mathcal{P}}$, let

$$\bar{t} = \rho^{\varphi}(\overline{P}).$$

Note that $\bar{t}$ is the minimum value of ρ^{φ} on its full domain. This implies that for any preference profile $P \in \mathcal{P}^n$, $\varphi(P)$ executes the cycles of $\tau(P)$ emerging from step 1 to step $\bar{t}-1$, that is, all cycles in the set $C_1 \cup C_2 \cup \cdots \cup C_{\bar{t}-1}$.

This also implies that $\varphi(\overline{P})$ executes all cycles in $C_1 \cup C_2 \cup \cdots \cup C_{\bar{t}-1}$, but not all cycles in $C_{\bar{t}}$. Let $C_{\bar{t}} \in C_{\bar{t}}$ be a cycle that is not executed by $\varphi(\overline{P})$ and $C_{\bar{t}}$ be given as follows:

$$i_1 \to \omega_{i_2} \to i_2 \to \omega_{i_3} \to \dots \to i_{k-1} \to \omega_{i_k} \to i_k \to \omega_{i_{k+1}} \to i_{k+1} = i_1.$$

Let $S = \{i_1, i_2, \dots, i_k\}$. Then, for each $i_s \in S$, $\tau_{i_s}(\overline{P}) = \omega_{i_{s+1}}$. Since $\varphi(\overline{P})$ does not execute $C_{\bar{t}}$, there exists at least one agent in this cycle whose assignment under $\varphi(\overline{P})$ is not the object that she points to in cycle $C_{\bar{t}}$. Furthermore, there exists at least one agent in this cycle whose assignment under $\varphi(\overline{P})$ is worse than the TTC allocation because $\varphi(\overline{P})$ executes all cycles in $C_1 \cup C_2 \cup \cdots \cup C_{\bar{t}-1}$ and $S = \{i_1, i_2, \dots, i_k\}$ are eliminated at step $\bar{t}$ of the TTC procedure. Without loss of generality, we suppose that $\varphi(\overline{P})$ assigns i_k an object ranked lower than ω_{i_1} under preference $\overline{P}_{i_k}$.[8]

To reach a contradiction, we construct two specific misrepresentations for each agent i_s in the above cycle that $\varphi(\overline{P})$ does not execute. Specifically, $P^*_{i_s}$ is obtained from $\overline{P}_{i_s}$ by moving ω_{i_s} up to a position right below $\omega_{i_{s+1}}$. The preference $P^{\dagger}_{i_s}$ is defined for $s \le k-2$ and is derived from $P^*_{i_s}$ by shifting ω_{i_k} up to a position right below $\omega_{i_{s+1}}$ and moving ω_{i_s} down to a position right below ω_{i_k}.

In order to complete the proof of Theorem 1, we introduce the following two lemmas. See Appendix A.1 and Appendix A.2 for the proof of these two lemmas.

Lemma 1 Let $P^1 = (P^*_{i_1}, P^*_{i_2}, \dots, P^*_{i_k}, \overline{P}_{N\setminus S})$. Then, for each $i_s \in S$, we have $\varphi_{i_s}(P^1) = \omega_{i_s}$.[9]

Lemma 2 Let $P^+ = (P^{\dagger}_{i_1}, P^{\dagger}_{i_2}, \dots, P^{\dagger}_{i_{k-2}}, P^*_{i_{k-1}}, P^*_{i_k}, \overline{P}_{N\setminus S})$. Then, for each $i_s \in S$, $\varphi_{i_s}(P^+) = \omega_{i_{s+1}} = \tau_{i_s}(P^+)$.

Conclusion of the Proof of Theorem 1

(1) Consider the profile $Q^{k-2} = (P^{\dagger}_{i_1}, P^{\dagger}_{i_2}, \dots, P^{\dagger}_{i_{k-3}}, P^*_{i_{k-2}}, P^*_{i_{k-1}}, P^*_{i_k}, \overline{P}_{N\setminus S})$. When TTC is run with profiles Q^{k-2} and P^+, the same sets of cycles are generated and executed in steps $1, 2, \dots, \bar{t}$. By the definition of $\bar{t}$, $\varphi(Q^{k-2})$ executes cycles in $C_1 \cup C_2 \cup \cdots \cup C_{\bar{t}-1}$. By the structure of Q^{k-2} given above and individual rationality of φ, we can obtain that: $\varphi_{i_k}(Q^{k-2}) \in \{\omega_{i_1}, \omega_{i_k}\}$, $\varphi_{i_{k-1}}(Q^{k-2}) \in \{\omega_{i_k}, \omega_{i_{k-1}}\}$, $\varphi_{i_{k-2}}(Q^{k-2}) \in \{\omega_{i_{k-1}}, \omega_{i_{k-2}}\}$, and for each $s \le k-3$, $\varphi_{i_s}(Q^{k-2}) \in \{\omega_{i_{s+1}}, \omega_{i_k}, \omega_{i_s}\}$. Since $P^*_{i_{k-2}}$ is a lower misrepresentation of $P^{\dagger}_{i_{k-2}}$ at $\varphi_{i_{k-2}}(P^+) = \omega_{i_{k-1}}$ (by Lemma 2) and φ is upper invariant, we have $\varphi_{i_{k-2}}(Q^{k-2}) = \omega_{i_{k-1}}$. Furthermore, by individual rationality of φ, for each $i_s \in S$, we have $\varphi_{i_s}(Q^{k-2}) = \varphi_{i_s}(P^+) = \omega_{i_{s+1}} = \tau_{i_s}(P^+)$.

(2) Consider the profile $Q^{k-3} = (P^{\dagger}_{i_1}, P^{\dagger}_{i_2}, \dots, P^{\dagger}_{i_{k-4}}, P^*_{i_{k-3}}, P^*_{i_{k-3}}, P^*_{i_{k-1}}, P^*_{i_k}, \overline{P}_{N\setminus S})$. By similar arguments in (1), for each $i_s \in S$, we have $\varphi_{i_s}(Q^{k-3}) = \varphi_{i_s}(P^+) = \omega_{i_{s+1}} = \tau_{i_s}(P^+)$.

(3) Consider the profile $Q^1 \equiv P^1 = (P^*_{i_1}, P^*_{i_2}, \dots, P^*_{i_k}, \overline{P}_{N\setminus S})$. As described above, we can iteratively define profiles $Q^{k-2}, Q^{k-3}, \dots, Q^1$. Using similar arguments, for profile $Q^1 \equiv P^1 = (P^*_{i_1}, P^*_{i_2}, \dots, P^*_{i_k}, \overline{P}_{N\setminus S})$, we find $\varphi_{i_s}(Q^1) = \omega_{i_{s+1}} = \tau_{i_s}(Q^1)$ for each $i_s \in S$, which contradicts Lemma 1.

Now we reach a contradiction and the proof of Theorem 1 is thus completed. □

$\overline{P}_{i_s}$	$P_{i_s}{}^*$	$P^{\dagger}_{i_s}$
$\vdots$	$\vdots$	$\vdots$
$\omega_{i_{s+1}}$	$\omega_{i_{s+1}}$	$\omega_{i_{s+1}}$
$\vdots$	ω_{i_s}	ω_{i_k}
ω_{i_s}	$\vdots$	ω_{i_s}
$\vdots$	$\vdots$	$\vdots$

Remark 2. Ekici and Sethuraman (2024) provided a short proof for Ekici (2024). It is important for us to clarify the relationship between their argument and ours. In fact, directly substituting upper invariance for strategy-proofness in the proof of their Theorem 1 is insufficient to establish Theorem 1 in our paper. Specifically, the conclusion in the fourth paragraph of their proof relies critically on the assumption of strategy-proofness. Using their notation, if strategy-proofness were replaced by upper invariance, we cannot establish that $\Pi_i(P)$ and $\Psi_i(P)$ are the only two acceptable objects for $i \in N_{\Pi}$. Moreover, Ekici and Sethuraman (2024) employed the minimal size method in their proof. We also attempted to use the same method to prove Theorem 1 of the current paper, but the attempt was unsuccessful. Therefore, we turned to the TTC similarity index to establish the conclusion of our Theorem 1.

Remark 3. Independence of axioms. To demonstrate the independence of the axioms in both Theorem 1 and the forthcoming Theorem 2, we consider the following three rules. The *serial dictatorship rule* satisfies upper invariance, endowments-swapping-proofness, and Pareto efficiency, but fails to satisfy individual rationality. The *no-trade rule* satisfies strategy-proofness and individual rationality, but fails to satisfy endowments-swapping-proofness, and pair-efficiency. The rule φ^3 defined in Example 3 given in the Online Appendix of the present paper satisfies individual rationality, endowments-swapping-proofness and Pareto efficiency (and hence pair-efficiency), but violates upper invariance.

4. Characterizing via endowments-swapping-proofness

In this section, we introduce another characterization of the TTC rule. This result refines the characterization proposed by Fujinaka and Wakayama (2018) by weakening strategy-proofness to upper invariance. Coreno and Feng (2025) had already refined Fujinaka and Wakayama (2018)'s characterization by replacing strategy-proofness with endowment-truncation-proofness. Their characterization and our result complement each other, since the axiom of upper invariance is independent of endowment-truncation-proofness.

Theorem 2. *A rule satisfies individual rationality, endowments-swapping-proofness, and upper invariance if and only if it is the TTC rule.*

Proof. It is obvious that τ satisfies individual rationality, endowments-swapping-proofness, and upper invariance.

Next, we proceed by contradiction to prove the "only if" part. Suppose that there is a rule φ that differs from the TTC rule τ

[8] This implies $k \ne 1$. Thus, $k \ge 2$.

[9] It is worth noting that Ekici (2024) obtained the same conclusion as Lemma 1 under the assumption of strategy-proofness. Since upper invariance is strictly weaker than strategy-proofness, Lemma 1 cannot be theoretically derived directly from Ekici (2024)'s result and thus requires a separate proof. However, we observe that the proof of Lemma 1 in this paper can be fully finished by replacing strategy-proofness with upper invariance in Ekici (2024)'s original argument. Therefore, full credit for the proof of Lemma 1 should be attributed to Ekici (2024), as his argument remains valid even when the stronger assumption of strategy-proofness is replaced by the weaker upper invariance.

but meets the requirements of individual rationality, endowments-swapping-proofness, and upper invariance.

Our proof is established based on the minimal size method, which is introduced by Sethuraman (2016) and developed in Ekici and Sethuraman (2024). For each economy $e = (P, \omega) \in \mathcal{P}^n \times \mathcal{A}$, its **size**, denoted by $s(e)$, is the total number of acceptable objects for agents. Formally, for each $e = (P, \omega)$,

$$s(P, \omega) = \sum_{i \in N} |\{h \in H \, : \, hR_i\omega_i\}|.$$

We call $e = (P, \omega)$ a **conflict economy** if $\varphi(e) \neq \tau(e)$. Then there is at least one conflict economy since φ and τ are different by assumption. Among all conflict economies, let $\mathcal{E}^s \subseteq \mathcal{E}$ be the set of economies with **minimal size**. Formally,

$$\mathcal{E}^s = \underset{e \in \{e' \in \mathcal{E} : \varphi(e') \neq \tau(e')\}}{\arg\min} s(e).$$

We denote a generic element of $\mathcal{E}^s$ by $e^* = (P^*, \omega^*)$. Thus, the rule φ selects the same allocation as the TTC rule for each economy whose size is strictly smaller than that of e^*. That is, for each $e \in \mathcal{E}$ such that $s(e) < s(e^*)$, $\varphi(e) = \tau(e)$.

All subsequent discussions are based on the minimal size conflict economy $e^* = (P^*, \omega^*)$. Next, we compare the allocations $\varphi(e^*)$ and $\tau(e^*)$ with respect to P^*. Let $\tau(e^*) = x$ and $\varphi(e^*) = y$. Define

$$N_x = \{i \in N \, : \, x_i P_i^* y_i\},$$

$$N_y = \{i \in N \, : \, y_i P_i^* x_i\}, \text{ and}$$

$$\overline{N} = N \backslash \{N_x \cup N_y\} = \{i \in N \, : \, x_i = y_i\}.$$

Since x is the TTC allocation and thus Pareto efficient, we know that N_x is nonempty, while N_y and $\overline{N}$ may be empty.

The following lemma presents some facts about the structure of P^*, $x = \tau(e^*)$ and $y = \varphi(e^*)$.

Lemma 3. *The conflict economy e^* with minimal size and the allocations $x = \tau(e^*)$ and $y = \varphi(e^*)$ satisfy:*

(1) for each $i \in N_x$, y_i is adjacent to x_i under P_i^* and $y_i = \omega_i^*$;
(2) for each $j \in N_y$, x_j is adjacent to y_j under P_j^* and $x_j = \omega_j^*$;
(3) agents in either N_x, N_y, or $\overline{N}$ are assigned one another's endowments, that is, $x_{N_x} = y_{N_x} = \omega_{N_x^*}$, $x_{N_y} = y_{N_y} = \omega^*_{N_y}$, and $x_{\overline{N}} = y_{\overline{N}} = \omega^*_{\overline{N}}$;
(4) the set N_y contains no agent, that is, $N_y = \emptyset$.

See Appendix A.3 for the proof of Lemma 3. Parts (1), (3) and (4) of Lemma 3 imply the following: The set N_x can be partitioned into agents in distinct cycles of the TTC algorithm under P^*. There exists a sequence of agents $N_x^* = \{i_1, i_2, \ldots, i_k, i_{k+1}\} \subseteq N_x, i_{k+1} = i_1$ (where $|N_x^*| = k$) such that for each $i_s \in \{i_1, i_2, \ldots, i_k\}$, her last two acceptable objects are, in order, $x_{i_s} = \omega^*_{i_{s+1}}$ and $\omega^*_{i_s}$. Agents in N_x^* form a cycle during the process of TTC algorithm. Note that N_x^* is not necessarily equivalent to N_x because N_x may contain more than one cycles of the TTC algorithm. Here, we assume that N_x^* only contains one cycle.

We also assume that the agents in the set N_x^* are eliminated at the earliest step among all agents in N_x by the TTC algorithm under $e^* = (P^*, \omega^*)$. Suppose agents in N_x^* are eliminated in step t^* of the TTC algorithm under $e^* = (P^*, \omega^*)$. The following Lemma presents some facts about the preferences of agents in N_x^*.

Lemma 4. *Each agent $i_s \in N_x^*$ finds $\omega^*_{N_x^* \backslash \{i_s, i_{s+1}\}}$ unacceptable.*

See Appendix A.4 for the proof of Lemma 4.

Conclusion of the Proof for Theorem 2

In summary, by Lemmas 3 and 4, the restriction of the economy $e^* = (P^*, \omega^*)$ to N_x^* is described below. As marked with boxes, under the economy e^*, the rule φ assigns all agent in N_x^* their endowments. Let $P^{*\prime}_{i_2}$ be the preference derived from $P^*_{i_2}$ by moving $\omega^*_{i_1}$ upwards immediately after $\omega^*_{i_3}$ while the relative ranking of the other objects remain the same.

$P_{i_1}^*$	$P_{i_2}^*$	$P_{i_3}^*$	$\cdots$	$P_{i_k}^*$	$P_{i_2}^{*\prime}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\omega^*_{i_2}$	$\omega^*_{i_3}$	$\omega^*_{i_4}$	$\vdots$	$\omega^*_{i_1}$	$\omega^*_{i_3}$
$\boxed{\omega^*_{i_1}}$	$\boxed{\omega^*_{i_2}}$	$\boxed{\omega^*_{i_3}}$	$\vdots$	$\boxed{\omega^*_{i_k}}$	$\omega^*_{i_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\omega^*_{i_1}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let $\omega^{i_1 i_2}$ be the endowment profile in which agent i_1 and i_2 swap their endowments under $e^* = (P^*, \omega^*)$, while the other agents' endowments remain the same as in e^*. Specifically, under the endowment profile $\omega^{i_1 i_2}$, the endowments of the agents in N_x^* can be described as follows.

$\omega^{i_1 i_2}_{i_1}$	$\omega^{i_1 i_2}_{i_2}$	$\cdots$	$\omega^{i_1 i_2}_{i_k}$
$\omega^*_{i_2}$	$\omega^*_{i_1}$	$\cdots$	$\omega^*_{i_k}$

By the definition of the TTC algorithm and Lemma 3, we have $k \neq 1$.

We claim that $|N_x^*| = k \neq 2$. Suppose not, then agents i_1 and i_2 prefer each other's endowment and would both be better off if they swap endowments, which contradicts endowments-swapping-proofness of φ.

Therefore, $|N_x^*| = k \geq 3$. Based on the economy $e^* = (P^*, \omega^*)$, construct two new economies $e^{i_1 i_2}$ and $\overline{e^{i_1 i_2}}$ as follows:

- Economy $e^{i_1 i_2}$ is obtained from e^* by swapping the endowments of agents i_1 and i_2 while keeping the preference profile P^* unchanged, i.e., $e^{i_1 i_2} = (P^*, \omega^{i_1 i_2})$;
- Economy $\overline{e^{i_1 i_2}}$ is obtained from $e^{i_1 i_2}$ by modifying agent i_2's preference to $P^{*\prime}_{i_2}$ while keeping the endowment profile $\omega^{i_1 i_2}$ unchanged, i.e., $\overline{e^{i_1 i_2}} = ((P^{*\prime}_{i_2}, P^*_{-i_2}), \omega^{i_1 i_2})$.

We now compare the allocations selected by the TTC rule and φ for the three economies: e^*, $e^{i_1 i_2}$, and $\overline{e^{i_1 i_2}}$.

By the definition of the TTC algorithm, $\tau(e^*) = \tau(e^{i_1 i_2}) = \tau(\overline{e^{i_1 i_2}})$.

By the definition of the size function, $s(e^{i_1 i_2}) \geq s(e^*) > s(\overline{e^{i_1 i_2}})$.[10] Since e^* is the minimal size conflict economy, we have $\tau(\overline{e^{i_1 i_2}}) = \varphi(\overline{e^{i_1 i_2}})$. Specifically, $\tau_{i_2}(\overline{e^{i_1 i_2}}) = \varphi_{i_2}(\overline{e^{i_1 i_2}}) = \omega^*_{i_3}$. Now, $P^*_{i_2}$ is a lower misrepresentation of $P^{*\prime}_{i_2}$ at $\varphi_{i_2}(\overline{e^{i_1 i_2}}) = \omega^*_{i_3}$. Then upper invariance of φ implies $\varphi_{i_2}(e^{i_1 i_2}) = \varphi_{i_2}(\overline{e^{i_1 i_2}}) = \omega^*_{i_3}$. Therefore, $\omega^*_{i_3} P^*_{i_2} \omega^*_{i_2}$ and $\varphi_{i_2}(e^*) = \omega^*_{i_2}$ imply

$$\varphi_{i_2}(e^{i_1 i_2}) P^*_{i_2} \varphi_{i_2}(e^*). \tag{1}$$

In addition, by individual rationality of φ, we have $\varphi_{i_1}(e^{i_1 i_2}) R^*_{i_1} \omega^*_{i_2}$. Therefore, $\omega^*_{i_2} P^*_{i_1} \omega^*_{i_1}$ and $\varphi_{i_1}(e^*) = \omega^*_{i_1}$ imply

$$\varphi_{i_1}(e^{i_1 i_2}) P^*_{i_1} \varphi_{i_1}(e^*). \tag{2}$$

The relationships (1) and (2) contradict endowments-swapping-proofness of φ. The proof of the "only if" part is thus completed. □

Remark 4. Fujinaka and Wakayama (2018) employ mathematical induction for the proof of their Theorem 1. Their argument remains valid if the assumption of strategy-proofness is replaced with upper invariance. Specifically, strategy-proofness is invoked only in Step 2 of the proof of their Lemma 1, and the conclusion remains valid even when strategy-proofness is replaced by upper invariance at that step. In the current paper, we offer a proof for Theorem 2 that does not rely

[10] Specifically, $s(e^*) = s(\overline{e^{i_1 i_2}}) + 1$.

on induction. On the one hand, this serves as an alternative proof of the main result in Fujinaka and Wakayama (2018). On the other hand, our approach employs the minimal-size method, which enables us to identify special structures in minimally conflicting preference profiles and offers a novel perspective for addressing similar problems.

Remark 5. This paper adopts the same proof technique as Ekici and Sethuraman (2024), namely the minimal-size method. Following Ekici and Sethuraman (2024), we partition agents into three groups based on the relative positions of their assignments under φ and τ. However, a key step in their proof shows that, for each agent i in a group, denoted N_x, only x_i and y_i are acceptable objects. This assumption critically depends on strategy-proofness and fails when strategy-proofness is replaced by upper invariance. As a result, although the proof technique is similar, Lemma 3 of the current paper is novel and cannot be obtained by directly adapting the arguments in Ekici and Sethuraman (2024).

Remark 6. Independence of axioms. The independence of axioms used in Theorem 2 has already been illustrated in Remark 3.

5. Discussion

5.1. Alternative incentive compatibility axioms

Both of the main characterization results in this paper rely on upper invariance. In fact, apart from upper invariance, there exist many other ways to weaken strategy-proofness. This naturally raises the question: if we replace strategy-proofness with these weaker axioms, do the existing characterization results still hold? In this section, we review several axioms that are weaker than strategy-proofness proposed in the literature and discuss their logical relationships. In the next subsection, we examine whether these weaker axioms can be used to characterize the TTC rule.

We need to first define a set of misrepresentations. Starting from the true preference of an agent, consider the following types of operations.

- For each $i \in N$ and $P_i, P_i' \in \mathcal{P}$, we say that P_i' is a **truncation misrepresentation** of P_i at $h \in H$ if any object that is ranked above h under P_i' has the same preference rank under P_i' as under P_i, that is,

  (i) $U(h, P_i') \subseteq U(h, P_i)$,
  (ii) $P_i(h') = P_i'(h'), \forall h' \in \widehat{U}(h, P_i')$.

- For each $i \in N$ and $P_i, P_i' \in \mathcal{P}$, we say that P_i' is a **Maskin misrepresentation** of P_i at $h \in H$ if any object that is ranked above h under P_i' is also ranked above h under P_i, that is,

  $U(h, P_i') \subseteq U(h, P_i)$.

- For each $i \in N$ and $P_i, P_i' \in \mathcal{P}$, we say that P_i' is a **drop misrepresentation** for P_i if one object is dropped to the bottom of agent i's preference, that is, there is an object $h \in H$ such that

  (i) for each $h' \in H \backslash \{h\}$, $h' P_i' h$,
  (ii) $P_i|_{H\backslash\{h\}} = P_i'|_{H\backslash\{h\}}$.

- For each $i \in N$ and $P_i, P_i' \in \mathcal{P}$, we say that P_i' is an **endowment-truncation misrepresentation** for P_i if the endowment is moved upward in the reported preference P_i', while the relative ranking of all other objects remains unchanged under P_i', that is,

  (i) $P_i'(\omega_i) < P_i(\omega_i)$,
  (ii) $P_i|_{H\backslash\{\omega_i\}} = P_i'|_{H\backslash\{\omega_i\}}$.

Using these misrepresentations, we are now ready to define several axioms that are weaker than strategy-proofness.

- Truncation invariance requires that if an agent reports a truncation misrepresentation, then her allocation remains unchanged. A rule φ satisfies **truncation invariance (TI)** if for every economy $e = (P, \omega)$, agent $i \in N$, and truncation misrepresentation P_i' at $\varphi_i(P, \omega)$,

  $\varphi_i(P, \omega) = \varphi_i((P_i', P_{-i}), \omega)$.

- Truncation-proofness requires that if an agent reports a truncation misrepresentation, then she cannot obtain a strictly better allocation. A rule φ satisfies **truncation-proofness (TP)** if for every economy $e = (P, \omega)$, agent $i \in N$, and truncation misrepresentation P_i' at $\varphi_i(P, \omega)$,

  $\varphi_i(P, \omega) R_i \varphi_i((P_i', P_{-i}), \omega)$.

- Maskin strategy-proofness requires that if an agent reports a Maskin misrepresentation, then she cannot obtain a strictly better allocation. A rule φ satisfies **Maskin strategy-proofness (MSP)** if for every economy $e = (P, \omega)$, agent $i \in N$, and Maskin misrepresentation P_i' at $\varphi_i(P, \omega)$,

  $\varphi_i(P, \omega) R_i \varphi_i((P_i', P_{-i}), \omega)$.

- Drop strategy-proofness requires that if an agent drops one object to the bottom of her preference list while keeping the relative ranking of all other objects unchanged, she cannot obtain a strictly better allocation. A rule φ satisfies **drop strategy-proofness (DSP)** if for every economy $e = (P, \omega)$, agent $i \in N$, and drop misrepresentation P_i' for P_i,

  $\varphi_i(P, \omega) R_i \varphi_i((P_i', P_{-i}), \omega)$.

- Upper-proofness requires that if an agent misreports only below their assigned object, then she cannot obtain a strictly better allocation. A rule φ satisfies **upper-proofness (UP)** if for every economy $e = (P, \omega)$, agent $i \in N$, and lower misrepresentation P_i' at $\varphi_i(P, \omega)$,

  $\varphi_i(P, \omega) R_i \varphi_i((P_i', P_{-i}), \omega)$.

- Endowment-truncation-proofness requires that if an agent moves her endowment upward in her reported preference without changing the relative ranking of the other objects, then they cannot obtain a strictly better allocation. A rule φ satisfies **endowment-truncation-proofness (ETP)** if for every economy $e = (P, \omega)$, agent $i \in N$, and endowment truncation misrepresentation P_i' for P_i,

  $\varphi_i(P, \omega) R_i \varphi_i((P_i', P_{-i}), \omega)$.

For incentive compatibility axioms defined above, including strategy-proofness (SP), drop strategy-proofness (DSP), truncation invariance (TI), Maskin strategy-proofness (MSP), endowment-truncation-proofness (ETP), upper invariance (UI), truncation-proofness (TP), and upper-proofness (UP), it is easy to check the logical relationship between each pair of them. Two types of logical relationships are identified: (1) two axioms are independent; (2) one axiom implies the other, but the latter does not imply the former. These relationships can be represented collectively in a diagram (See Fig. 1). The diagram is constructed as follows. If two axioms are logically independent, no arrow is drawn between them. If axiom A implies axiom B, but B does not imply A, this is represented either by an arrow from A to B, or by a sequence of arrows which form a path from A to B. We give a detailed demonstration for these relationships in the online Appendix of this paper.

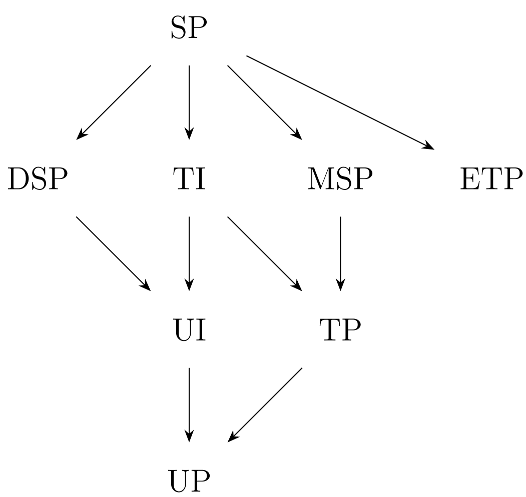

Fig. 1. Logical relationships among incentive compatibility axioms.

5.2. Other characterizations of the TTC rule

Based on the conclusion of Theorem 1 and the logical relationships among truncation invariance, upper invariance, and drop strategy-proofness revealed by Fig. 1, we can directly derive the following new characterizations of the TTC rule.

Corollary 1 (*Characterizations Via Pair-Efficiency*)**.** *The TTC rule can be characterized by either of the following two groups of axioms:*
(i) individual rationality, pair-efficiency, and drop strategy-proofness;
(ii) individual rationality, pair-efficiency, and truncation invariance.

Based on the conclusion of Theorem 2 and the logical relationships among truncation invariance, upper invariance, and drop strategy-proofness revealed by Fig. 1, we can directly derive the following new characterizations of the TTC rule.

Corollary 2 (*Characterizations Via Endowments-Swapping-Proofness*)**.** *The TTC rule can be characterized by either of the following two groups of axioms:*
(i) individual rationality, endowments-swapping-proofness, and drop strategy-proofness;
(ii) individual rationality, endowments-swapping-proofness, and truncation invariance.

Respecting-improvement requires that if the ranking of a given agent's endowment improves at a set of agents' preferences, while keeping their preferences over other objects and other agents' preferences unchanged, the given agent must be weakly better off as a result of the improvement. Formally, let $i \in N$, $e = (P, \omega) \in \mathcal{E}$, and $e' = (P', \omega) \in \mathcal{E}$. We say that e' is an improvement for agent i with respect to e if the following conditions are satisfied: (1)$P_i = P'_i$, and (2) for each $j \neq i$, $P_j(\omega_i) \geq P'_j(\omega_i)$ and $P_j|_{H\setminus\{\omega_i\}} = P'_j|_{H\setminus\{\omega_i\}}$. A rule φ **respects improvements (RI)** (or satisfies respecting-improvement) if for each agent $i \in N$, each economy $e = (P, \omega) \in \mathcal{E}$, and each improvement for i at e denoted as $e' = (P', \omega) \in \mathcal{E}$, we have $\varphi_i(e') R_i \varphi_i(e)$.

Klaus et al. (2025) show that the TTC rule is the unique strategy-proof rule that is pair-efficient and respects improvements. Their Lemma 1 states that if a rule is pair-efficient and respects improvements then it is individual rational. Building on this result, our Theorems 1 and 2 directly imply three alternative characterizations of the TTC rule.

Corollary 3 (*Characterizations Via Respecting-Improvement*)**.** *The TTC rule can be characterized by each of the following three groups of axioms:*
(i) pair-efficiency, respecting-improvement, and upper invariance;
(ii) pair-efficiency, respecting-improvement, and drop strategy-proofness;
(iii) pair-efficiency, respecting-improvement, and truncation invariance.

5.3. On the logical relationship between endowments-swapping-proofness and pair-efficiency

Theorems 1 and 2 characterize the TTC rule using two distinct sets of axioms. The difference between these two sets lies in the use of pair-efficiency and endowments-swapping-proofness. Intuitively, pair-efficiency concerns the absence of beneficial pairwise trades *after* the reallocation of endowments, whereas endowments-swapping-proofness focuses on preventing beneficial trades *before* reallocation. Consequently, it is natural to seek a deeper understanding of the intrinsic relationship between the two axioms. This section offers a tentative exploration of that relationship.[11]

Endowments-swapping-proofness and pair-efficiency are logically independent by definition. They are not mutually exclusive, as the TTC rule and the serial dictatorship rule satisfy both properties. In addition, we find the following relationships between the two properties under different circumstances.

(1) **Independence under Individual Rationality.** The logical independence between endowments-swapping-proofness and pair-efficiency persists even when individual rationality is imposed. For instance, the NSE rule defined in Example 8 of Fujinaka and Wakayama (2018) satisfies pair-efficiency and individual rationality but violates endowments-swapping-proofness. The NEF rule defined in Example 5 of Fujinaka and Wakayama (2018) satisfies endowments-swapping-proofness and individual rationality but violates pair-efficiency.
(2) **Independence under Strategy-proofness.** Even when strategy-proofness is imposed, the two properties remain independent. The SD^1 rule defined in Fujinaka and Wakayama (2018)'s example 6 is a rule that satisfies pair-efficiency and strategy-proofness but violates endowments-swapping-proofness. The constant rule is a rule that satisfies endowments-swapping-proofness and strategy-proofness but violates pair-efficiency.
(3) **Logical Implication under the Combination of Individual Rationality and Endowment-truncation-proofness.** Coreno and Feng (2025)'s Theorem 1 and Example 1 demonstrate that pair-efficiency is strictly weaker than endowments-swapping-proofness for rules that satisfy both individual rationality and endowment-truncation-proofness.
(4) **Equivalence under the Combination of Individual Rationality and Upper Invariance.** Our theorems demonstrate that pair-efficiency and endowments-swapping-proofness become equivalent for rules that satisfy both individual rationality and upper invariance (or drop strategy-proofness, or truncation invariance).

6. Conclusion

This paper offers some new characterizations of the TTC rule in the housing market, refining existing results in the literature. These findings deepen our understanding of the TTC rule. The main findings are summarized in Table 1 below.

We conclude this paper by suggesting some directions for future research. First, it is possible to extend our model to some more general settings beyond the classic housing market model. A key question is whether the two groups of axioms presented in our paper can still characterize the generalized TTC rule in such general environments. For instance, in a model where agents can hold and consume multiple objects and have lexicographic preferences over bundles of objects, Altuntaş et al. (2023) show that individual rationality, Pareto efficiency, and upper invariance (or drop strategy-proofness) characterize the

[11] We thank one referee for the valuable suggestion to explore the relationship between the two axioms.

Table 1
Existing characterizations of the TTC rule.

Literature	IR	PE	SP	Replace PE		Replace SP				Replace IR
				ESP	PaE	UI	DSP	ETP	TI	RI
Ma (1994)	✓	✓	✓							
Fujinaka and Wakayama (2018)	✓		✓	✓						
Ekici (2024)	✓		✓		✓					
Altuntaş et al. (2023) A[a]	✓	✓					✓			
Altuntaş et al. (2023) B[b]	✓	✓				✓				
Coreno and Feng (2025)	✓			✓				✓		
Theorem 1	✓				✓	✓				
Corollary 1	✓				✓		✓			
	✓				✓				✓	
Theorem 2	✓			✓		✓				
Corollary 2	✓			✓			✓			
	✓			✓					✓	
Klaus et al. (2025)			✓		✓					✓
Corollary 3					✓	✓				✓
					✓		✓			✓
					✓				✓	✓

The notation "✓" in a cell means that the axiom is satisfied in the corresponding literature. The notation IR, PE, SP, ESP, PaE, UI, DSP, ETP, TI, RI respectively refer to the axioms of individual rationality, Pareto efficiency, strategy-proofness, endowments-swapping-proofness, pair-efficiency, upper invariance, drop strategy-proofness, endowment-truncation-proofness, truncation invariance, and respecting-improvement.
[a] Altuntaş et al. (2023) A refers to Corollary 3 in their paper.
[b] Altuntaş et al. (2023) B refers to Corollary 4 in their paper.

generalized TTC rule. It remains to be seen whether Theorems 1 and 2 of the current paper continue to hold in this context.

Second, we may consider alternative refinement of the characterization of the TTC rule by further weakening the underlying axioms. For example, strategy-proofness can be weakened by adopting the individual version of weak Maskin monotonicity, as defined by Kojima and Manea (2010).[12] Likewise, some characterizations of the TTC rule rely on group strategy-proofness. A promising direction for further research is to examine whether this stronger condition can be replaced by weaker variants. In particular, weak Maskin monotonicity is strictly weaker than group strategy-proofness, and it would be worthwhile to investigate whether existing characterizations remain valid when group strategy-proofness is substituted with this weaker condition.

CRediT authorship contribution statement

Yajing Chen: Writing – review & editing, Writing – original draft, Supervision, Methodology, Formal analysis, Conceptualization. **Zhenhua Jiao:** Writing – review & editing, Funding acquisition, Formal analysis. **Chenfeng Zhang:** Software, Methodology, Conceptualization. **Luosai Zhang:** Writing – review & editing, Conceptualization, Formal analysis, Methodology.

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Declaration of competing interest

The authors have no competing interests to declare that are relevant to the content of this article.

Appendix A. Proof of lemmas

In this Appendix, we present the proof of the lemmas stated in the main text.

A.1. Proof of lemma 1

Proof. (i) Consider the profile $P^k = (P^*_{i_k}, \overline{P}_{N\setminus\{i_k\}})$.

We first claim that $\varphi_{i_k}(P^k) = \omega_{i_k}$. Suppose on the contrary $\varphi_{i_k}(P^k) \neq \omega_{i_k}$, then by individual rationality of φ, $\varphi_{i_k}(P^k) \in U(P^*_{i_k}, \omega_{i_1})$. Now, $\overline{P}_{i_k}$ is a lower misrepresentation of $P^*_{i_k}$ at $\varphi_{i_k}(P^k)$. Since φ satisfies upper invariance, we have $\varphi_{i_k}(\overline{P}) = \varphi_{i_k}(P^k) \in U(P^*_{i_k}, \omega_{i_{k+1}}) = U(\overline{P}_{i_k}, \omega_{i_{k+1}})$, which contradicts the fact that $\varphi(\overline{P})$ assigns i_k an object worse than $\omega_{i_{k+1}}$ at $\overline{P}$.

When TTC runs with profiles P^k and $\overline{P}$, the same sets of cycles arise, and the same cycles are executed at steps $1, 2, \ldots, \bar{t}$. Since $\bar{t}$ is the minimum value of ρ^φ on its domain, $\varphi(P^k)$ executes the cycles in $C_1 \cup C_2 \cup \cdots \cup C_{\bar{t}-1}$. Among the remaining objects after excluding these cycles, i_{k-1}'s favorite object at $\overline{P}_{i_{k-1}}$ is ω_{i_k}. Therefore, $\varphi_{i_k}(P^k) = \omega_{i_k}$ implies that $\varphi(P^k)$ assigns i_{k-1} an object worse than ω_{i_k} at $\overline{P}_{i_{k-1}}$.

(ii) Consider the profile $P^{k-1} = (P^*_{i_{k-1}}, P^*_{i_k}, \overline{P}_{N\setminus\{i_{k-1}, i_k\}})$.

By applying a similar argument in (i), it is easy to verify that $\varphi_{i_{k-1}}(P^{k-1}) = \omega_{i_{k-1}}$,[13] and $\varphi(P^{k-1})$ assigns i_{k-2} an object worse than $\omega_{i_{k-1}}$ at $\overline{P}_{i_{k-2}}$.

(iii) Consider the profile $P^1 = (P^*_{i_1}, P^*_{i_2}, \ldots, P^*_{i_k}, P_{N\setminus S})$.

As described above, we can iteratively define the profiles $P^{k-2}, P^{k-3}, \ldots, P^1$. By similar arguments, for profile $P^1 = (P^*_{i_1}, P^*_{i_2}, \ldots, P^*_{i_k}, \overline{P}_{N\setminus S})$, we find: $\varphi_{i_1}(P^1) = \omega_{i_1}$. When TTC runs with profiles P^1 and $\overline{P}$, the same sets of cycles arise, and the same cycles are executed, at steps $1, 2, \ldots, \bar{t}$. Since $\bar{t}$ is the minimizer of ρ^φ on its domain, $\varphi(P^k)$ executes the cycles in $C_1 \cup C_2 \cup \cdots \cup C_{\bar{t}-1}$. After excluding these cycles, for each $i_s \in S$, $\varphi_{i_s}(P^1) \in \{\omega_{i_{s+1}}, \omega_{i_s}\}$. Then, by individual rationality of φ, $\varphi_{i_1}(P^1) = \omega_{i_1}$ implies that for each $i_s \in S$, $\varphi_{i_s}(P^1) = \omega_{i_s}$. The proof of Lemma 1 is completed. □

[12] Formally, a rule φ satisfies individual weak Maskin monotonicity if for each economy $e = (P, \omega)$, each agent $i \in N$, and each Maskin misrepresentation P'_i of P_i at $\varphi_i(P, \omega)$, we have $\varphi_i((P'_i, P_{-i}), \omega) R'_i \varphi_i(P, \omega)$. Suppose a rule φ satisfies strategy-proofness. For each economy $e = (P, \omega)$, each agent $i \in N$, and each Maskin misrepresentation P'_i of P_i at $\varphi_i(P, \omega)$, note that P_i is also a misrepresentation of P'_i. By strategy-proofness of φ, it follows that $\varphi((P'_i, P_{-i}), \omega)\ R'_i\ \varphi(P, \omega)$. Hence, by definition, φ satisfies individual weak Maskin monotonicity.

[13] This implies $k \neq 2$. Otherwise, the individual rationality of φ implies $\varphi_{i_k}(P^{k-1}) = \omega_{i_k}$. However this cannot be true since i_k and i_{k-1} prefer one another's assigned objects to their own assignments while φ is pair-efficient.

A.2. Proof of lemma 2

Proof. Let $P^+ = (P^\dagger_{i_1}, P^\dagger_{i_2}, \ldots, P^\dagger_{i_{k-2}}, P^*_{i_{k-1}}, P^*_{i_k}, \overline{P}_{N\setminus S})$, then the restriction of P^+ to $S = \{i_1, i_2, \ldots, i_k\}$ has the following structure:

$P^\dagger_{i_1}$	$P^\dagger_{i_2}$	$\cdots$	$P^\dagger_{i_{k-2}}$	$P^*_{i_{k-1}}$	$P^*_{i_k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ω_{i_2}	ω_{i_3}	$\vdots$	$\omega_{i_{k-1}}$	ω_{i_k}	ω_{i_1}
ω_{i_k}	ω_{i_k}	$\vdots$	ω_{i_k}	$\omega_{i_{k-1}}$	ω_{i_k}
ω_{i_1}	ω_{i_2}	$\vdots$	$\omega_{i_{k-2}}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

When TTC runs with profiles P^+ and $\overline{P}$, the same sets of cycles arise and are executed at steps $1, 2, \ldots, \bar{t}$. By the definition of $\bar{t}$, $\varphi(P^+)$ executes the cycles in $C_1 \cup C_2 \cup \cdots \cup C_{\bar{t}-1}$. By the structure of P^+ given above and individual rationality of φ, we can obtain that: $\varphi_{i_k}(P^+) \in \{\omega_{i_1}, \omega_{i_k}\}$, $\varphi_{i_{k-1}}(P^+) \in \{\omega_{i_k}, \omega_{i_{k-1}}\}$, and for each $s \leq k-2$, $\varphi_{i_s}(P^+) \in \{\omega_{i_{s+1}}, \omega_{i_k}, \omega_{i_s}\}$.

(i) We first claim that $\varphi_{i_k}(P^+) = \omega_{i_1}$. Otherwise, $\varphi_{i_k}(P^+) = \omega_{i_k}$. Then by individual rationality of φ we have $\varphi_{i_1}(P^+) = \omega_{i_1}$ (for each $s \in S$, $\varphi_{i_s}(P^+) = \omega_{i_s}$), contradicting pair-efficiency of φ.

(ii) We then claim that $\varphi_{i_{k-1}}(P^+) = \omega_{i_k}$. Suppose instead that $\varphi_{i_{k-1}}(P^+) = \omega_{i_{k-1}}$. We discuss this in two cases:

Case 1: $\varphi_{i_{k-2}}(P^+) = \omega_{i_k}$. In this case, we would have $\varphi_{i_{k-1}}(P^+) = \omega_{i_{k-1}}$ and $\varphi_{i_{k-2}}(P^+) = \omega_{i_k}$, which directly contradicts pair-efficiency of φ.

Case 2: for some $s < k-2$, $\varphi_{i_s}(P^+) = \omega_{i_k}$. Then, by the structure of P^+ and individual rationality of φ, we have $\varphi_{i_{s+1}}(P^+) = \omega_{i_{s+1}}$. Thus, $\varphi_{i_s}(P^+) = \omega_{i_k}$ and $\varphi_{i_{s+1}}(P^+) = \omega_{i_{s+1}}$ contradict the pair-efficiency of φ.

Therefore, we conclude that $\varphi_{i_{k-1}}(P^+) = \omega_{i_k}$.

(iii) Finally, we claim that for each $s \leq k-2$, $\varphi_{i_s}(P^+) = \omega_{i_{s+1}}$. This follows from the conclusions of parts (i) and (ii), together with individual rationality of φ. The proof of Lemma 2 is completed. □

A.3. Proof of Lemma 3

Proof (*1*)**.** We will prove this part in two steps by contradiction.

Step 1, we show that $y_i = \omega^*_i$. Suppose this is not true for some $i \in N_x$. Then, by assumption, we have $x_i P^*_i y_i P^*_i \omega^*_i$. Construct a new preference order for agent i, denoted $\overline{Q}_i$, by modifying P^*_i as follows: (1) move ω^*_i upwards by one preference ranking, i.e., $\overline{Q}_i(\omega^*_i) = P^*_i(\omega^*_i) - 1$, and (2) preserve the relative ranking of all objects except ω^*_i, i.e., $\overline{Q}_i|_{H\setminus\{\omega^*_i\}} = P^*_i|_{H\setminus\{\omega^*_i\}}$. Let $\overline{Q} = (\overline{Q}_i, P^*_{-i})$ denote the new preference profile. By the construction of $\overline{Q}$, $s(\overline{Q}, \omega^*) < s(P^*, \omega^*)$. Since (P^*, ω^*) is the conflict economy with minimal size, we have $\varphi_i(\overline{Q}, \omega^*) = \tau_i(\overline{Q}, \omega^*) = x_i$. Now, P^*_i is a lower misrepresentation of $\overline{Q}_i$ at x_i, but $\varphi_i(\overline{Q}, \omega^*) = x_i \neq y_i = \varphi_i(P^*, \omega^*)$, which contradicts the assumption that φ is upper invariant.

Step 2, we show that y_i is adjacent to x_i. Suppose this is not true for some $i \in N_x$. Construct a new preference order for agent i, denoted $\overline{Q}_i$, by modifying P^*_i as follows: (1) move $y_i = \omega^*_i$ upwards immediately after x_i, i.e., $Q_i(y_i) = P^*_i(x_i) + 1$, and (2) preserve the relative ranking of all objects except y_i, i.e., $Q_i|_{H\setminus\{y_i\}} = P^*_i|_{H\setminus\{y_i\}}$. Let $Q = (Q_i, P^*_{-i})$ denote the new preference profile. By the construction of Q, $s(Q, \omega^*) < s(P^*, \omega^*)$. By the definition of size function, we have $\varphi_i(Q, \omega^*) = \tau_i(Q, \omega^*) = x_i$. Now, P^*_i is a lower misrepresentation of Q_i at x_i, but $\varphi_i(Q, \omega^*) = x_i \neq y_i = \varphi_i(P^*, \omega^*)$, which contradicts the assumption that φ is upper invariant.

(2) The proof is similar to that of part (1) and is thus omitted.

(3) This is a natural corollary of part (1) and (2).

(4) Suppose $N_y \neq \emptyset$. Then, $N_y \cup \overline{N} \neq \emptyset$. By part (3), under the allocation y, $N_y \cup \overline{N}$ forms an allocation weakly dominates x. This contradicts the assumption that x is in the core. The proof of Lemma 3 is completed. □

A.4. Proof of Lemma 4

Proof. By the definition of TTC algorithm, for each $i_s \in N^*_x$, the upper contour set of $P^*_{i_s}$ at x_{i_s}, i.e., $U(x_{i_s}, P^*_{i_s})$ (possibly empty) are all objects eliminated during step 1 to $t^* - 1$ of the TTC algorithm. Since N^*_x is eliminated in step t^* of the TTC algorithm under P^*, it follows that $\omega^*_{N^*_x} \cap U(P^*_{i_s}, x_{i_s}) = \emptyset$. Moreover, because $\omega^*_{N^*_x\setminus\{i_s, i_{s+1}\}} \subset \omega^*_{N^*_x}$, we naturally have $\omega^*_{N^*_x\setminus\{i_s, i_{s+1}\}} \cap U(P^*_{i_s}, x_{i_s}) = \emptyset$. By part (1) of Lemma 3, for each $i_s \in N^*_x$, after eliminating $U(P^*_{i_s}, x_i)$, the only two acceptable objects are $\omega^*_{i_s}$ and $\omega^*_{i_{s+1}}$. Therefore, for each $i_s \in N^*_x$, all objects in $\omega^*_{N^*_x\setminus\{i_s, i_{s+1}\}}$ are unacceptable. The proof of Lemma 4 is completed. □

Appendix B. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jmateco.2025.103190.

Data availability

No data was used for the research described in the article.

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